The Method of Moderation

Preprint, compiled May 14, 2026

Christopher D. Carroll [1, 2*], **Alan Lujan** [1, 2†], **Karsten Chipeniuk**[3], **Kiichi Tokuoka**[4], **and Weifeng Wu**[5]

[1]Johns Hopkins University
[2]Econ-ARK
[3]Reserve Bank of New Zealand
[4]Japanese Ministry of Finance
[5]Fannie Mae

Abstract

In a risky world, a pessimist assumes the worst will happen. Someone who ignores risk altogether is an optimist. Consumption decisions are mathematically simple for both the pessimist and the optimist because both behave as if they live in a riskless world. A consumer who is a realist (that is, who wants to respond optimally to risk) faces a much more difficult problem, but (under standard conditions) will choose a level of spending somewhere between that of the pessimist and the optimist. We use this fact to redefine the space in which the realist searches for optimal consumption rules. The resulting solution accurately represents the numerical consumption rule over the entire interval of feasible wealth values with remarkably few computations.

***K*eywords** Dynamic Stochastic Optimization, Consumption-Saving Models, Numerical Methods

1 Introduction

Solving a consumption-saving problem using numerical methods requires the modeler to choose how to represent a policy function. In the stochastic case, where analytical solutions are generally not available, a common approach is to use low-order polynomial splines that exactly match the function at a finite set of gridpoints, and then to assume that interpolated or extrapolated versions of that spline represent the function well at the continuous infinity of unmatched points. Carroll [1] developed the endogenous gridpoints method (EGM), which has become a standard tool for computing consumption at gridpoints determined endogenously using the Euler equation.

Unfortunately, this endogenous gridpoints solution is not very well-behaved outside the original range of gridpoints (though other common solution methods are no better outside their own predefined ranges). Figure 1 demonstrates the point. The figure shows the approximated precautionary component of saving, the amount by which the realist consumes less than an optimist with the same expected income path. Theory proves that precautionary saving is always positive, yet the linearly extrapolated numerical approximation eventually predicts negative precautionary saving. However, in the presence of uncertainty, the consumption-saving rule must be evaluated outside *any* prespecified grid. This is because large positive shock realizations push next period's assets for a sufficiently wealthy individual beyond the grid boundaries.

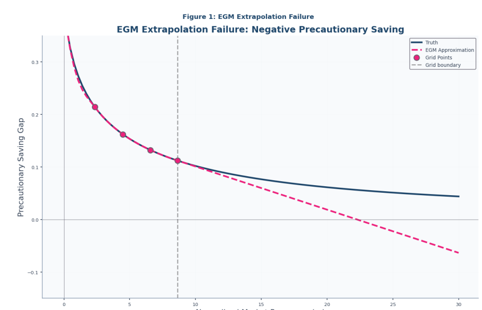

Figure 1: For Large Enough Resources m, Predicted Precautionary Saving is Negative (Oops!)

This error cannot be fixed by extending the upper gridpoint. While extrapolation techniques can prevent this from being fatal, the problem is often dealt with using inelegant methods whose implications for accuracy are difficult to gauge.

This paper argues that, in the standard consumption problem, a better approach is to rely upon the fact that without uncertainty, the optimal consumption function has a simple analytical solution. The key insight is that, under standard assumptions, the consumer who faces an uninsurable labor income risk will consume less than a consumer with the same path for expected income but who does not perceive any uncertainty. Following Leland [2], Sandmo [3], and Kimball [4], the 'realist' consumer, who *does* perceive the risks, will engage in 'precautionary saving,' so the perfect foresight riskless solution provides an upper bound to the solution that will actually be optimal. A lower bound is provided by the behavior of a consumer who has the subjective belief that the future level of income will be the worst that it can possibly be. This consumer, too, behaves according to the convenient analytical perfect foresight solution, but his certainty is that of a pessimist perfectly confident in his pessimism.

We build on bounds for the consumption function and limiting MPCs established by Stachurski and Toda [5], Ma et al. [6], Carroll [7], and Ma and Toda [8] in buffer-stock theory. Using results from Carroll and Shanker [9], we show how to use these bounds to constrain the shape and characteristics of the solution to the 'realist' problem characterized by Carroll [10].

Imposition of these constraints clarifies and speeds the solution of the realist's problem. For comparison, we use the endogenous gridpoints method [1] as our benchmark, which computes consumption at gridpoints determined endogenously using the Euler equation.

After showing how to use the method in the baseline case, we show how to refine it to encompass an even tighter theoretical bound, and how to extend it to solve a problem in which the consumer faces both labor income risk and rate-of-return risk.

2 THE REALIST'S PROBLEM

Consider a consumer who correctly perceives all risks. The consumer has CRRA utility over consumption c with risk aversion parameter $\rho > 0$:

$$\mathbf{u}(c) = \begin{cases} \frac{c^{1-\rho}}{1-\rho} & \text{if } \rho \neq 1 \\ \log c & \text{if } \rho = 1. \end{cases} \tag{1}$$

This utility function satisfies prudence ($\mathbf{u}''' > 0$), which induces precautionary saving. The consumer maximizes expected lifetime utility, discounted by the time preference factor β:

$$\max_{c_t, a_t} \mathbf{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathbf{u}(c_{t+n}) \right] \tag{2}$$

subject to $c_t + a_t = m_t$, where m denotes market resources and a denotes assets. We focus on resources of the form $m_{t+1} = a_t \mathrm{R}_{t+1} + y_{t+1}$, where R_{t+1} denotes the interest rate, and y_{t+1} labor income. Initially we take R_{t+1} to be deterministic, and relax this later.

While our method can be adapted to a range of stochastic labor income processes, to fix ideas we suppose income evolves via the Friedman-Muth process (Friedman [11] distinguished permanent from transitory income; Muth [12] provided the stochastic framework). That is, $y_{t+1} = p_{t+1}\xi_{t+1}$ where p denotes permanent labor income and ξ_{t+1} a transitory component. Permanent income growth is given by $p_{t+1} = p_t \mathcal{G}_{t+1}$, where the gross growth factor $\mathcal{G}_{t+1} = G_{t+1}\psi_{t+1}$. Here G_{t+1} is the deterministic permanent income growth factor, while ψ_{t+1} are permanent shocks with mean unity and bounded support $[\underline{\psi}, \bar{\psi}]$ where $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$. Transitory shocks ξ_{t+1} take value 0 with probability $\wp > 0$ (unemployment) or $\theta_{t+1}/(1-\wp)$ otherwise, with $\mathbf{E}_t[\theta_{t+1}] = 1$.

This problem can be rewritten (see Carroll [13] for a proof) in a more convenient form in which choice and state variables are normalized by the level of permanent income, e.g., replacing m_t with m_t/p_t. When that is done, the Bellman equation for the value function $\mathbf{v}$ of the transformed problem is

$$\mathbf{v}_t(m_t) = \max_{c_t, a_t} \left(\mathbf{u}(c_t) + \beta \mathbf{E}_t[\mathcal{G}_{t+1}^{1-\rho} \mathbf{v}_{t+1}(m_{t+1})] \right) \tag{3}$$

subject to the normalized transition $m_{t+1} = (\mathrm{R}/\mathcal{G}_{t+1})a_t + \xi_{t+1}$, with Euler equation $\mathbf{u}'(c_t) = \beta \mathrm{R} \mathbf{E}_t[\mathcal{G}_{t+1}^{-\rho}\mathbf{u}'(c_{t+1})]$.

Carroll and Shanker [9] gives conditions for a finite solution of the problem with a Friedman-Muth process. Consider the case of time-invariant G_t, ψ_t, and R_t, and define the absolute patience factor $\Phi \equiv (\beta \mathrm{R})^{1/\rho}$. Then a finite solution requires: (i) finite-value-of-autarky condition (FVAC) $0 < \beta G^{1-\rho}\mathbf{E}[\psi^{1-\rho}] < 1$, (ii) absolute-impatience condition (AIC) $\Phi < 1$, (iii) return-impatience condition (RIC) $\Phi/\mathrm{R} < 1$, (iv) growth-impatience condition (GIC) $\Phi/G < 1$, and (v) finite-human-wealth condition (FHWC) $G/\mathrm{R} < 1$. These patience conditions ensure the consumption bounds and limiting MPCs used in our method.

For expositional simplicity, in the numerical results that follow, we assume $\rho \neq 1$ and set $G = 1$, $\psi = 1$ (no permanent income growth or shocks). The figures display the next-to-last period of a finite horizon problem, where the last-period analytical solution provides exact boundary conditions. The method extends naturally to the infinite-horizon case and to general parameter configurations.

3 THE METHOD OF MODERATION

3.1 The Optimist, the Pessimist, and the Realist

As a preliminary to our solution, define $\bar{h}$[3] as end-of-period human wealth (the present discounted value of future labor income) for a perfect foresight version of the problem of a 'risk optimist:' a consumer who believes with perfect confidence that the shocks will always take their expected value of 1, $\xi_{t+n} = \mathbf{E}[\xi] = 1 \; \forall \, n > 0$. The solution to a perfect foresight problem of this kind takes the form

$$\bar{\mathbf{c}}(m) = (m + \bar{h})\underline{\boldsymbol{\kappa}} \tag{4}$$

for a constant minimal marginal propensity to consume $\underline{\boldsymbol{\kappa}}$.[4] We similarly define $\underline{h}$[5] as 'minimal human wealth,' the present discounted value of labor income if the shocks were to take on their worst value in every future period $\xi_{t+n} = \underline{\xi} \; \forall \, n > 0$. We refer to a consumer who expects to encounter this sequence of shocks as a 'pessimist'. Their consumption decision rule is given by

$$\underline{\mathbf{c}}(m) = (m + \underline{h})\underline{\boldsymbol{\kappa}}. \tag{5}$$

We will call a 'realist' the consumer who correctly perceives the true probabilities of the future risks and optimizes accordingly.

A first useful point is that, for the realist, a lower bound for the level of market resources is the natural borrowing constraint $\underline{m} = -\underline{h}$ derived by Aiyagari [14] and Huggett [15], because if m equalled this value then there would be a positive finite chance (however small) of receiving $\xi_{t+n} = \underline{\xi}$ in every future period, which would require the consumer to set c to zero in order to guarantee that the intertemporal budget constraint

[3] Setting $\xi_{t+n} = 1$ (the optimist's assumption), human wealth in infinite-horizon is $\bar{h} = G/(\mathrm{R}-G)$ if $\mathrm{R} > G$. When $G = 1$, $\bar{h} = 1/(\mathrm{R}-1)$.

[4] The MPC of the perfect foresight consumer: infinite-horizon $\underline{\boldsymbol{\kappa}} = 1 - \Phi/\mathrm{R}$.

[5] Setting $\xi_{t+n} = \underline{\xi} \; \forall \, n > 0$, minimal human wealth is $\underline{h} = \underline{\xi}G/(\mathrm{R}-G)$ if $\mathrm{R} > G$. When $\underline{\xi} = 0$, $\underline{h} = 0$.

holds. Since consumption of zero yields infinite marginal utility, Zeldes [16] and Deaton [17] show that the solution to the realist consumer's problem is not well defined for values of $m \leq \underline{m}$, and the limiting value of the realist's c is zero as $m \downarrow \underline{m}$ (established in Carroll and Shanker [9]).

It is convenient to define 'excess' market resources as the amount by which actual resources exceed the lower bound, and 'excess' human wealth as the amount by which mean expected human wealth exceeds guaranteed minimum human wealth:

$$\begin{aligned} \Delta m &= m + \overbrace{\underline{h}}^{=-\underline{m}} \\ \Delta h &= \bar{h} - \underline{h}. \end{aligned} \tag{6}$$

We now rewrite the optimal consumption rules for the two perfect foresight problems in terms of excess resources and human wealth. The 'pessimist' perceives human wealth to be equal to its minimum feasible value $\underline{h}$ with certainty, and so consumes a constant fraction of current excess resources

$$\underline{\mathbf{c}}(m) = \Delta m \underline{\boldsymbol{\kappa}}. \tag{7}$$

The 'optimist,' on the other hand, pretends that there is no uncertainty about future income, and therefore consumes the same fraction out of current excess resources *and* excess human wealth

$$\bar{\mathbf{c}}(m) = (\Delta m + \Delta h)\underline{\boldsymbol{\kappa}}. \tag{8}$$

3.2 *The Moderation Ratio*

The pessimist expects the worst possible income in every future period with certainty, and must finance all future consumption through saving; this generates the most aggressive saving behavior. A realist, by contrast, can reoptimize as uncertainty resolves each period, so need not prepare for the worst with certainty. At the same time, the adverse outcome remains possible, so even a wealthy realist maintains some precautionary saving: a sufficiently well-off individual is nearly (but never completely) self-insured, and thus mostly smooths consumption like the optimist. The realist therefore consumes strictly more than the pessimist but strictly less than the optimist at every wealth level, as shown in Figure 2.

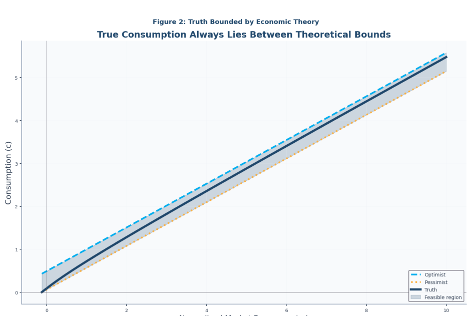

Figure 2: Moderation Illustrated: $\underline{\mathbf{c}} < \grave{\mathbf{c}} < \bar{\mathbf{c}}$

The proof is more difficult than might be imagined, but the necessary work is done in Carroll and Shanker [9] so we will take the proposition as a fact:

$$\underline{\mathbf{c}}(\underline{m} + \Delta m) < \hat{\mathbf{c}}(\underline{m} + \Delta m) < \bar{\mathbf{c}}(\underline{m} + \Delta m) \tag{9}$$

Subtracting $\underline{\mathbf{c}}(\underline{m} + \Delta m)$ in each of these inequalities and using equations (7) and (8) gives

$$\begin{aligned} 0 < \quad & \hat{\mathbf{c}}(\underline{m} + \Delta m) - \underline{\mathbf{c}}(\underline{m} + \Delta m) && < \Delta h \underline{\boldsymbol{\kappa}} \\ 0 < \quad & \underbrace{\left(\frac{\hat{\mathbf{c}}(\underline{m} + \Delta m) - \underline{\mathbf{c}}(\underline{m} + \Delta m)}{\Delta h \underline{\boldsymbol{\kappa}}} \right)}_{\equiv \boldsymbol{\omega}} && < 1 \end{aligned} \tag{10}$$

where the fraction in the middle of the last inequality is the moderation ratio measuring how close the realist's consumption is to the optimist's behavior (the numerator is the gap between the realist and pessimist) relative to the maximum possible gap between optimist and pessimist. When $\boldsymbol{\omega} = 0$, the realist behaves like the pessimist (maximum precautionary saving); when $\boldsymbol{\omega} = 1$, the realist behaves like the optimist (no precautionary saving). Carroll and Kimball [18] and Carroll and Shanker [9] establish that under bounded shocks, $\boldsymbol{\omega} \in (0, 1)$ strictly for all $m > \underline{m}$. Defining $\boldsymbol{\mu} = \log \Delta m$ (which can range from $-\infty$ to ∞), the object in the middle of the last inequality is

$$\boldsymbol{\omega}(\boldsymbol{\mu}) \equiv \left(\frac{\hat{\mathbf{c}}(\underline{m} + e^{\boldsymbol{\mu}}) - \underline{\mathbf{c}}(\underline{m} + e^{\boldsymbol{\mu}})}{\Delta h \underline{\boldsymbol{\kappa}}} \right), \tag{11}$$

and we now define

$$\begin{aligned} \boldsymbol{\chi}(\boldsymbol{\mu}) &= \log \left(\frac{\boldsymbol{\omega}(\boldsymbol{\mu})}{1 - \boldsymbol{\omega}(\boldsymbol{\mu})} \right) \\ &= \log(\boldsymbol{\omega}(\boldsymbol{\mu})) - \log(1 - \boldsymbol{\omega}(\boldsymbol{\mu})) \end{aligned} \tag{12}$$

which has the virtue that it is *asymptotically linear* in the limit as $\boldsymbol{\mu}$ approaches $+\infty$.[6] Since $\boldsymbol{\omega} \in (0, 1)$, the ratio $\boldsymbol{\omega}/(1 - \boldsymbol{\omega})$ is the odds ratio, and $\boldsymbol{\chi}$ is the log odds ratio, the same transformation that underpins logit regression in econometrics. The logit maps $\boldsymbol{\omega} \in (0, 1)$ to $\boldsymbol{\chi} \in (-\infty, \infty)$ with inverse sigmoid $\boldsymbol{\omega} = 1/(1 + \exp(-\boldsymbol{\chi}))$; log maps $(m - \underline{m}) \in (0, \infty)$ to $\boldsymbol{\mu} \in (-\infty, \infty)$.

Given $\boldsymbol{\chi}$, the consumption function can be recovered from

$$\hat{\mathbf{c}} = \underline{\mathbf{c}} + \overbrace{\frac{1}{1 + \exp(-\boldsymbol{\chi})}}^{=\boldsymbol{\omega}} \Delta h \underline{\boldsymbol{\kappa}}. \tag{13}$$

Thus, the method of moderation (MoM) is to calculate $\boldsymbol{\chi}$ at the points $\boldsymbol{\mu}$ corresponding to the log of the Δm points defined above, and then using these to construct an interpolating

[6] Under the GIC, $\boldsymbol{\chi}(\boldsymbol{\mu})$ is asymptotically linear with slope $\alpha \geq 0$ as $\boldsymbol{\mu} \to +\infty$. We extrapolate $\boldsymbol{\chi}$ linearly using the boundary slope, preserving $\boldsymbol{\omega} \in (0, 1)$ and hence $\underline{\mathbf{c}} < \grave{\mathbf{c}} < \bar{\mathbf{c}}$ throughout the extrapolation domain.

approximation $\grave{\chi}$ from which we indirectly obtain our approximated consumption rule $\grave{\mathbf{c}}$ (an approximation to the true $\hat{\mathbf{c}}$) by substituting $\grave{\chi}$ for χ in equation (13).

Because this method relies upon the fact that the problem is easy to solve if the decision maker has unreasonable views (either in the optimistic or the pessimistic direction), and because the correct solution is always between these immoderate extremes, we call our solution procedure the 'method of moderation.'

Results are shown in Figure 3; a reader with very good eyesight might be able to detect the barest hint of a discrepancy between the Truth and the Approximation at the far right-hand edge of the figure, a stark contrast with the calamitous divergence evident in Figure 1.

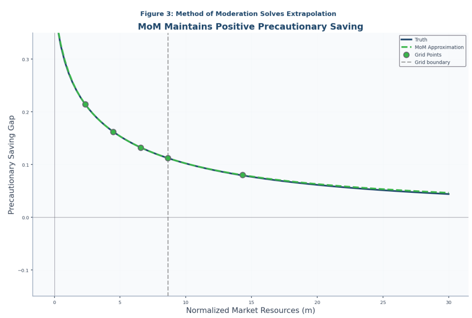

Figure 3: Extrapolated $\grave{\mathbf{c}}$ Constructed Using the Method of Moderation

4 EXTENSIONS

4.1 A Tighter Upper Bound

The method described above does not guarantee that the approximated consumption function respects the constraint $\hat{\mathbf{c}}(m) < \bar{\boldsymbol{\kappa}}\Delta m$, where $\bar{\boldsymbol{\kappa}}$ is the MPC at the natural borrowing constraint. Near the constraint, the optimist's bound $\bar{\mathbf{c}}$ becomes loose because it is calibrated to the low MPC that prevails at high wealth. A tighter upper bound for low-wealth consumers eliminates this slack.

Carroll and Shanker [9] derives an explicit formula for this maximal MPC: $\bar{\boldsymbol{\kappa}} = 1 - \wp^{1/\rho}(\Phi/\mathrm{R})$ where $\wp$ is the unemployment probability derived by Carroll and Toche [19], extending the explicit limiting MPC formulas established in buffer-stock theory by Ma and Toda [8]. Strict concavity of the consumption function implies $\hat{\mathbf{c}}(m) < \bar{\boldsymbol{\kappa}}\Delta m$ for low wealth, while the optimist's bound $\hat{\mathbf{c}}(m) < \bar{\mathbf{c}}(m) = (\Delta m + \Delta h)\underline{\boldsymbol{\kappa}}$ is tighter for high wealth.

As shown in Figure 4, the two upper bounds intersect at the cusp point:

$$m^* = -\underline{h} + \frac{\underline{\boldsymbol{\kappa}}(\bar{h} - \underline{h})}{\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}} \tag{14}$$

This intersection occurs in the feasible region since $\bar{\boldsymbol{\kappa}} > \underline{\boldsymbol{\kappa}}$ under the stated conditions (the MPC is highest when wealth is lowest).

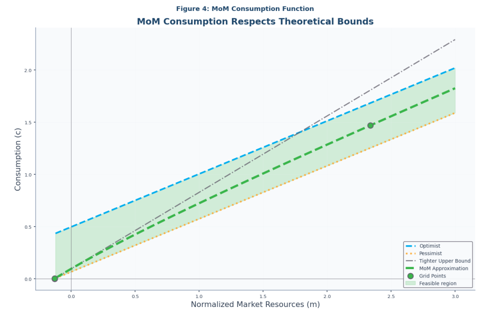

Figure 4: A Tighter Upper Bound

Table 1: Maximum absolute approximation errors by interval. Orders of magnitude in parentheses.

Method	$[m_0, m_1]$	$[m_1, m_2]$	$[m_2, m_3]$	$[m_3, m_4]$	$[m_4, \overline{m}]$
EGM	8.6(-3)	1.8(-4)	2.5(-5)	7.3(-6)	1.1(-1)
MoM	2.9(-3)	4.3(-6)	6.6(-7)	1.3(-7)	2.4(-3)

For $m < m^*$, define the low-resource moderation ratio using the tighter bound:

$$\grave{\boldsymbol{\omega}}(\boldsymbol{\mu}) = \frac{\hat{\mathbf{c}}(\underline{m} + e^{\boldsymbol{\mu}})e^{-\boldsymbol{\mu}} - \underline{\boldsymbol{\kappa}}}{\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}} \tag{15}$$

Since $e^{-\boldsymbol{\mu}} = 1/\Delta m$, the right-hand side equals $(\hat{\mathbf{c}}/\Delta m - \underline{\boldsymbol{\kappa}})/(\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}})$, which lies in $(0, 1)$ for $m \in (\underline{m}, m^*]$: the lower bound is the optimist's MPC and the upper bound is the maximal MPC, with strict inequality at the upper end following from $m < m^*$ (see (21) in the Appendix). This ratio measures how far consumption per unit of wealth exceeds the optimist's MPC, relative to the maximum possible excess. Applying the logit transformation and interpolating as before yields consumption satisfying both upper bounds throughout.

For computational robustness, construct a three-piece approximation: below the cusp using the tight bound, near the cusp using Hermite interpolation [20] (see Section 4.3) matching levels and slopes at adjacent gridpoints, above the cusp using the original optimist bound. This ensures continuous, differentiable consumption functions respecting all theoretical constraints.

The MoM also contributes to literature which aims to improve the precision of dynamic stochastic optimization solutions, such as Chipeniuk [21]. Table 1 demonstrates the accuracy gains obtained with the method between each pair of grid points m_j, m_{j+1}, as well as for the extrapolation of the consumption function to $\overline{m} = 30$. Displayed is the maximum absolute difference between the true consumption function and each approximation, evaluated on a dense subgrid of each interval. In each region the MoM produces an approximation which is more than an order of magnitude more accurate than the basic EGM.

4.2 Value Function

Often it is useful to know the value function as well as the consumption rule. Fortunately, many of the tricks used when solving for the consumption rule have a direct analogue in approximation of the value function. Define the inverse value function transformation

$$\bar{\mathbf{\Lambda}} = ((1-\rho)\bar{\mathbf{v}})^{1/(1-\rho)} \tag{16}$$

which under perfect foresight equals $(\Delta m + \Delta h)\underline{\boldsymbol{\kappa}}^{-\rho/(1-\rho)}$ (linear in market resources). Analogously to the consumption moderation ratio, we define a value moderation ratio $\hat{\mathbf{\Omega}}$ that measures the proximity of the realist's inverse value to the optimist's (see equation (26) in the Appendix for the precise definition). The logit transformation $\hat{\boldsymbol{X}}$ is applied as before. Interpolate $\hat{\boldsymbol{X}}$ at gridpoints and invert to obtain

$$\hat{\mathbf{v}} = \mathbf{u}(\hat{\mathbf{\Lambda}}). \tag{17}$$

4.3 Hermite Interpolation

The numerical accuracy of the method of moderation depends critically on the quality of function approximation between gridpoints [22]. Our bracketing approach complements work that bounds numerical errors in dynamic economic models [23]. Although linear interpolation that matches the level of $\hat{\mathbf{c}}$ at the gridpoints is simple, Hermite interpolation [20] offers a considerable advantage.

The moderation ratio derivative measures how quickly the realist approaches the optimist as resources increase. Differentiating (11) with respect to $\boldsymbol{\mu}$ we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\mu}} = \frac{\Delta m(\partial \hat{\mathbf{c}}/\partial m - \underline{\boldsymbol{\kappa}})}{\underline{\boldsymbol{\kappa}}\Delta h}. \tag{18}$$

Rearranging this yields a moderation form for the marginal propensity to consume:

$$\frac{\partial \hat{\mathbf{c}}}{\partial m} = (1-\boldsymbol{\eta})\,\underline{\boldsymbol{\kappa}} + \boldsymbol{\eta}\,\bar{\boldsymbol{\kappa}} \tag{19}$$

where

$$\boldsymbol{\eta} = \frac{\underline{\boldsymbol{\kappa}}}{\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}} \cdot \frac{\Delta h}{\Delta m} \cdot \partial\boldsymbol{\omega}/\partial\boldsymbol{\mu}. \tag{20}$$

Carroll and Shanker [9] guarantees $\underline{\boldsymbol{\kappa}} \le \partial\hat{\mathbf{c}}/\partial m \le \bar{\boldsymbol{\kappa}}$ at gridpoints where the Euler equation holds, so $\boldsymbol{\eta} \in [0,1]$ and the expression above is indeed a convex combination of $\underline{\boldsymbol{\kappa}}$ and $\bar{\boldsymbol{\kappa}}$. At very high wealth, $\boldsymbol{\eta} \to 0$ and the MPC approaches $\underline{\boldsymbol{\kappa}}$; near the borrowing constraint, $\boldsymbol{\eta} \to 1$ and the MPC approaches $\bar{\boldsymbol{\kappa}}$.

For Hermite interpolation, compute $\partial\boldsymbol{\omega}/\partial\boldsymbol{\mu}$ at gridpoints, then derive $\partial\boldsymbol{\chi}/\partial\boldsymbol{\mu} = \partial\boldsymbol{\omega}/\partial\boldsymbol{\mu}/[\boldsymbol{\omega}(1-\boldsymbol{\omega})]$ for slope data. By matching both the level and the derivative of $\hat{\mathbf{c}}$ at the gridpoints, where the derivative is obtained from the envelope condition [24, 25] together with the EGM Euler equation, the interpolated consumption rule satisfies the Euler equation exactly at each solved gridpoint. These techniques extend naturally to the value function approximation.

For monotone cubic Hermite schemes [20, 26, 27], theoretical slopes may be adjusted to enforce monotonicity [28]. The Fritsch-Carlson algorithm modifies slopes at local extrema, while Fritsch-Butland uses harmonic mean weighting. Both preserve the shape-preserving property essential for consumption functions that must be strictly increasing.

4.4 Stochastic Rate of Return

For i.i.d. returns with $\log \mathbf{R} \sim \mathcal{N}(r + \pi - \sigma_{\mathbf{r}}^2/2, \sigma_{\mathbf{r}}^2)$,[7] Samuelson [29], Merton [30, 31] showed that for a consumer without labor income (or with perfectly forecastable labor income) the consumption function is linear, with an MPC $= 1 - (\beta\mathbf{E}[\mathbf{R}^{1-\rho}])^{1/\rho}$. See Carroll [32], Benhabib and Bisin [33], Chipeniuk et al. [34] for extensions. The pessimist and the optimist face certain income but the same stochastic return, so the Merton-Samuelson result applies to both and their consumption functions remain linear. The realist faces both labor income and return risk, and the moderation ratio captures their combined precautionary response. Substitute this stochastic MPC for $\underline{\boldsymbol{\kappa}}$ throughout our formulas; the optimist's and pessimist's human-wealth definitions $\bar{h}$ and $\underline{h}$ are adjusted in parallel by replacing R with $\mathbf{E}[\mathbf{R}]$ in their PDV formulas. As Figure 5 shows, consumption remains bounded between the pessimist and the optimist, each of which consume slightly less in the face of return uncertainty.

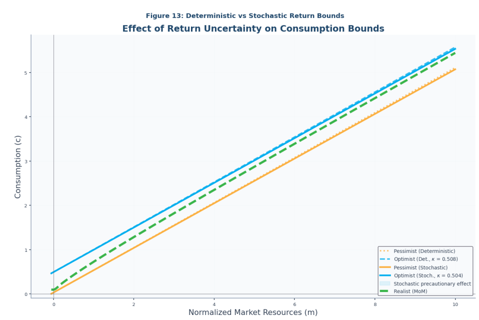

Figure 5: Effect of Return Uncertainty on Consumption Bounds

5 CONCLUSION

The method proposed here is not universally applicable. For example, the method cannot be used for problems for which upper and lower bounds to the 'true' solution are not known. But many problems do have obvious upper and lower bounds, and in those cases (as in the consumption example used in the paper), the method may result in substantial improvements in accuracy and stability of solutions. The method of moderation is efficient because the transformed moderation ratio is better-behaved than consumption, requiring fewer gridpoints. As the accuracy results in Table 1 confirm, these gains compound: the MoM approximation produces errors more than an order of magnitude smaller than the basic EGM across all grid intervals, including the extrapolation region where standard methods fail most severely.

[7] Here r is the log risk-free rate and π is the equity premium (the expected excess log return). This parametrization ensures $\mathbf{E}[\mathbf{R}] = \exp(r+\pi)$, so that increasing $\sigma_{\mathbf{r}}^2$ constitutes a mean-preserving spread of the level of the return.

A Appendix: Mathematical Details

A.1 Patience Conditions Details

The patience conditions listed in the main text have clear economic interpretations. The FVAC $0 < \beta G^{1-\rho}\mathbf{E}[\psi^{1-\rho}] < 1$ ensures that autarky (saving nothing, consuming all income each period) yields finite expected discounted utility, guaranteeing the consumer values resources. The AIC $\Phi < 1$ prevents indefinite consumption deferral by ensuring the marginal utility of current consumption exceeds the discounted marginal utility of future consumption under certainty. The RIC $\Phi/\mathsf{R} < 1$ ensures asset growth is slower than the patience-adjusted discount rate, preventing unbounded wealth accumulation. The GIC $\Phi/G < 1$ ensures consumption grows slower than permanent income, establishing a target wealth ratio. The FHWC $G/\mathsf{R} < 1$ ensures the present value of future labor income is finite. Together, these conditions partition parameter space into regions with qualitatively different behavior: buffer-stock saving with a target wealth ratio (all conditions hold), perpetual borrowing (AIC fails), or unbounded wealth growth (GIC fails but RIC holds) [10, 13, 9].

A.2 Human Wealth Formulas

The optimist's human wealth (assuming $\xi_{t+n} = 1\ \forall\ n > 0$) can be computed three ways: backward recursion $\bar{h}_T = 0$, $\bar{h}_t = (G/\mathsf{R})(1 + \bar{h}_{t+1})$; forward sum $\bar{h}_t = \sum_{n=1}^{T-t}(G/\mathsf{R})^n$; or infinite-horizon $\bar{h} = G/(\mathsf{R} - G)$ when $\mathsf{R} > G$. With $G = 1$, $\bar{h} = 1/(\mathsf{R} - 1)$.

The pessimist's human wealth (assuming $\xi_{t+n} = \underline{\xi}\ \forall\ n > 0$) follows similarly: backward recursion $\underline{h}_T = 0$, $\underline{h}_t = (G/\mathsf{R})(\underline{\xi} + \underline{h}_{t+1})$; forward sum $\underline{h}_t = \underline{\xi}\sum_{n=1}^{T-t}(G/\mathsf{R})^n$; or infinite-horizon $\underline{h} = \underline{\xi}G/(\mathsf{R} - G)$. When $\underline{\xi} = 0$ (unemployment), $\underline{h} = 0$.

A.3 Marginal Propensity to Consume Formulas

The minimal MPC (perfect foresight consumer with horizon $T - t$) has three forms [7]: backward recursion $\underline{\boldsymbol{\kappa}}_t = \underline{\boldsymbol{\kappa}}_{t+1}/(\underline{\boldsymbol{\kappa}}_{t+1} + \Phi/\mathsf{R})$ with $\underline{\boldsymbol{\kappa}}_T = 1$; forward sum $\underline{\boldsymbol{\kappa}}_t = (\sum_{n=0}^{T-t}(\Phi/\mathsf{R})^n)^{-1}$; or infinite-horizon $\underline{\boldsymbol{\kappa}} = 1 - \Phi/\mathsf{R} = 1 - (\mathsf{R}\beta)^{1/\rho}/\mathsf{R}$.

The maximal MPC [19] satisfies backward recursion $\bar{\boldsymbol{\kappa}}_t = \bar{\boldsymbol{\kappa}}_{t+1}/(\bar{\boldsymbol{\kappa}}_{t+1} + \wp^{1/\rho}\Phi/\mathsf{R})$ with $\bar{\boldsymbol{\kappa}}_T = 1$; forward sum $\bar{\boldsymbol{\kappa}}_t = (\sum_{n=0}^{T-t}(\wp^{1/\rho}\Phi/\mathsf{R})^n)^{-1}$; or infinite-horizon $\bar{\boldsymbol{\kappa}} = 1 - \wp^{1/\rho}(\Phi/\mathsf{R})$.

A.4 Cusp Point Calculation

The two upper bounds intersect at the cusp point m^* where

$$\begin{aligned} \left(\Delta m^* + \Delta h\right)\underline{\boldsymbol{\kappa}} &= \bar{\boldsymbol{\kappa}}\,\Delta m^* \\ \Delta m^* &= \frac{\underline{\boldsymbol{\kappa}}\,\Delta h}{\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}} \\ m^* &= -\underline{h} + \frac{\underline{\boldsymbol{\kappa}}\left(\bar{h} - \underline{h}\right)}{\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}}, \end{aligned} \tag{21}$$

where $\Delta m^* \equiv m^* - \underline{m} > 0$ since $\bar{\boldsymbol{\kappa}} > \underline{\boldsymbol{\kappa}}$. For $m \in (\underline{m}, m^*]$, the tighter upper bound yields

$$\begin{aligned} \Delta m\underline{\boldsymbol{\kappa}} < &\quad \hat{\mathbf{c}}(\underline{m} + \Delta m) &< \bar{\boldsymbol{\kappa}}\Delta m \\ 0 < &\quad \hat{\mathbf{c}}(\underline{m} + \Delta m) - \Delta m\underline{\boldsymbol{\kappa}} &< \Delta m(\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}}) \\ 0 < &\quad \left(\frac{\hat{\mathbf{c}}(\underline{m}+\Delta m) - \Delta m\underline{\boldsymbol{\kappa}}}{\Delta m(\bar{\boldsymbol{\kappa}} - \underline{\boldsymbol{\kappa}})}\right) &< 1. \end{aligned} \tag{22}$$

This motivates the definition of the low-resource moderation ratio as in (15).

A.5 Value Function Derivation

Under perfect foresight, consumption grows at constant rate equal to the absolute patience factor Φ: $\mathbf{c}_{t+n} = \mathbf{c}_t\Phi^n$. The present discounted value of consumption satisfies $\mathrm{PDV}_t^T(\mathbf{c}) = \sum_{n=0}^{T-t}\beta^n\mathbf{c}_t\Phi^n = \mathbf{c}_t\sum_{n=0}^{T-t}(\Phi/\mathsf{R})^n$, where we use $\beta\Phi^{1-\rho} = \Phi/\mathsf{R}$. Dividing by consumption yields the PDV-to-consumption ratio $\mathcal{C}_t^T = \mathrm{PDV}_t^T(\mathbf{c})/\mathbf{c}_t = \sum_{n=0}^{T-t}(\Phi/\mathsf{R})^n = \underline{\boldsymbol{\kappa}}_t^{-1}$, which is unchanged for normalized variables. Defining $\mathcal{C} \equiv \lim_{T\to\infty}\mathcal{C}_t^T$, this yields the key identity $\mathcal{C} = \underline{\boldsymbol{\kappa}}^{-1}$, connecting the infinite-horizon PDV-to-consumption ratio to the minimal MPC.

The optimist's value function satisfies

$$\begin{aligned} \bar{\mathbf{v}}_{T-1}(m_{T-1}) &\equiv \mathbf{u}(c_{T-1}) + \beta\mathbf{u}(c_T) \\ &= \mathbf{u}(c_{T-1})\left(1 + \beta\Phi^{1-\rho}\right) \\ &= \mathbf{u}(c_{T-1})\left(1 + \Phi/\mathsf{R}\right) \\ &= \mathbf{u}(c_{T-1})\mathcal{C}_{T-1}^T \end{aligned} \tag{23}$$

The infinite horizon expression becomes

$$\begin{aligned} \bar{\mathbf{v}}(m) &= \mathbf{u}(\bar{\mathbf{c}}(m))\mathcal{C} \\ &= \mathbf{u}(\bar{\mathbf{c}}(m))\underline{\boldsymbol{\kappa}}^{-1} \\ &= \mathbf{u}((\Delta m + \Delta h)\underline{\boldsymbol{\kappa}})\underline{\boldsymbol{\kappa}}^{-1} \\ &= \mathbf{u}(\Delta m + \Delta h)\underline{\boldsymbol{\kappa}}^{-\rho}. \end{aligned} \tag{24}$$

This can be transformed as

$$\begin{aligned} \bar{\boldsymbol{\Lambda}} &\equiv ((1-\rho)\bar{\mathbf{v}})^{1/(1-\rho)} \\ &= c\,\mathcal{C}^{1/(1-\rho)} \\ &= (\Delta m + \Delta h)\underline{\boldsymbol{\kappa}}^{-\rho/(1-\rho)}. \end{aligned} \tag{25}$$

For the realist's problem, we define $\hat{\boldsymbol{\Lambda}} = ((1-\rho)\hat{\mathbf{v}}(m))^{1/(1-\rho)}$. Using the bounds $\underline{\boldsymbol{\Lambda}} < \hat{\boldsymbol{\Lambda}} < \bar{\boldsymbol{\Lambda}}$, we define

$$\hat{\boldsymbol{\Omega}}(\boldsymbol{\mu}) = \left(\frac{\hat{\boldsymbol{\Lambda}}(\underline{m} + e^{\boldsymbol{\mu}}) - \underline{\boldsymbol{\Lambda}}(\underline{m} + e^{\boldsymbol{\mu}})}{\Delta h\underline{\boldsymbol{\kappa}}\,\mathcal{C}^{1/(1-\rho)}}\right) \tag{26}$$

and the logit-transformed counterpart:

$$\begin{aligned} \hat{\boldsymbol{X}}(\boldsymbol{\mu}) &= \log\left(\frac{\hat{\boldsymbol{\Omega}}(\boldsymbol{\mu})}{1 - \hat{\boldsymbol{\Omega}}(\boldsymbol{\mu})}\right) \\ &= \log(\hat{\boldsymbol{\Omega}}(\boldsymbol{\mu})) - \log(1 - \hat{\boldsymbol{\Omega}}(\boldsymbol{\mu})) \end{aligned} \tag{27}$$

Inverting these approximations yields

$$\hat{\mathbf{\Lambda}} = \underline{\mathbf{\Lambda}} + \overbrace{\left(\frac{1}{1+\exp(-\hat{\boldsymbol{X}})}\right)}^{=\hat{\mathbf{\Omega}}} \Delta h \underline{\boldsymbol{\kappa}} \mathcal{C}^{1/(1-\rho)} \tag{28}$$

from which the value function approximation is $\hat{\mathrm{v}} = \mathrm{u}(\hat{\mathbf{\Lambda}})$.

A.6 I.I.D. Stochastic Returns: MPC derivation

The fact that a linear consumption function with an MPC $= 1 - (\beta\mathbf{E}[\mathbf{R}^{1-\rho}])^{1/\rho}$ satisfies the Euler equation with i.i.d. returns and no labor income can be derived by the method of undetermined coefficients. In particular, assume that $\bar{\mathrm{c}}(m) = m\underline{\boldsymbol{\kappa}}$, with a time-independent MPC $\underline{\boldsymbol{\kappa}}$ to be determined. Substituting this into the Euler equation, we have

$$\begin{aligned} 1 &= \beta\mathbf{E}_t\left[\mathbf{R}_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\right] \\ &= \beta\mathbf{E}_t\left[\mathbf{R}_{t+1}\left(\frac{m_{t+1}}{m_t}\right)^{-\rho}\right] \end{aligned} \tag{29}$$

where the second equality uses the assumed form of the consumption function. Since there is no labor income, $m_{t+1} = \mathbf{R}_{t+1}(m_t - c_t)$. Substituting this into the above we obtain

$$1 = \beta\mathbf{E}_t\left[\mathbf{R}_{t+1}\left(\mathbf{R}_{t+1}(1-\underline{\boldsymbol{\kappa}})\right)^{-\rho}\right] \tag{30}$$

Solving for $\underline{\boldsymbol{\kappa}}$ and recalling that returns are i.i.d. gives $\underline{\boldsymbol{\kappa}} = 1 - (\beta\mathbf{E}[\mathbf{R}^{1-\rho}])^{1/\rho}$.

In the particular case of lognormal returns, the MPC can be written in closed form. The moment generating function (MGF) for lognormal returns provides the key formula. For $\log\mathbf{R} \sim \mathcal{N}(\mu, \sigma^2)$, the MGF is $\mathbf{E}[e^{sX}] = \exp(\mu s + \sigma^2 s^2/2)$ where $X = \log\mathbf{R}$. Setting $s = 1-\rho$ and $\mu = r + \pi - \sigma_{\mathbf{r}}^2/2$ yields[8]

$$\mathbf{E}[\mathbf{R}^{1-\rho}] = \exp\left((1-\rho)\left(r + \pi - \frac{\sigma_{\mathbf{r}}^2}{2}\right) + \frac{(1-\rho)^2\sigma_{\mathbf{r}}^2}{2}\right). \tag{31}$$

Simplifying the variance terms: $(1-\rho)^2\sigma_{\mathbf{r}}^2/2 - (1-\rho)\sigma_{\mathbf{r}}^2/2 = (1-\rho)[(1-\rho)-1]\sigma_{\mathbf{r}}^2/2 = -\rho(1-\rho)\sigma_{\mathbf{r}}^2/2$, giving the final form

$$\mathbf{E}[\mathbf{R}^{1-\rho}] = \exp\left((1-\rho)\left(r + \pi - \rho\sigma_{\mathbf{r}}^2/2\right)\right). \tag{32}$$

REFERENCES

[1] Christopher D Carroll. The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3):312–320, 2006. doi: 10.1016/j.econlet.2005.09.013.

[2] Hayne E. Leland. Saving and Uncertainty: The Precautionary Demand for Saving. *Quarterly Journal of Economics*, 82(3):465–473, 1968. doi: 10.2307/1879518.

[3] Agnar Sandmo. The Effect of Uncertainty on Saving Decisions. *Review of Economic Studies*, 37(3):353–360, 1970. doi: 10.2307/2296725.

[4] Miles S. Kimball. Precautionary Saving in the Small and in the Large. *Econometrica*, 58(1):53–73, 1990. doi: 10.2307/2938334.

[5] John Stachurski and Alexis Akira Toda. An Impossibility Theorem for Wealth in Heterogeneous-Agent Models with Limited Heterogeneity. *Journal of Economic Theory*, 182:1–24, 2019. doi: 10.1016/j.jet.2019.04.001. Provides an explicit linear lower bound on consumption; Corrigendum 2020.

[6] Qingyin Ma, John Stachurski, and Alexis Akira Toda. The Income Fluctuation Problem and the Evolution of Wealth. *Journal of Economic Theory*, 187:105003, 2020. doi: 10.1016/j.jet.2020.105003.

[7] Christopher D. Carroll. Precautionary Saving and the Marginal Propensity to Consume Out of Permanent Income. *Journal of Monetary Economics*, 56(6):780–790, 2009. doi: 10.1016/j.jmoneco.2009.06.016.

[8] Qingyin Ma and Alexis Akira Toda. A Theory of the Saving Rate of the Rich. *Journal of Economic Theory*, 192:105193, 2021. doi: 10.1016/j.jet.2021.105193.

[9] Christopher D. Carroll and Akshay Shanker. Theoretical Foundations of Buffer Stock Saving (Revise and Resubmit, Quantitative Economics). Revise and Resubmit, Quantitative Economics, 6 2024. URL https://llorracc.github.io/BufferStockTheory/BufferStockTheory.pdf.

[10] Christopher D. Carroll. Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *Quarterly Journal of Economics*, 112(1):1–55, 1997. doi: 10.1162/003355397555109.

[11] Milton Friedman. *A Theory of the Consumption Function*. Princeton University Press, Princeton, NJ, 1957. doi: 10.1515/9780691188485.

[12] John F. Muth. Optimal Properties of Exponentially Weighted Forecasts. *Journal of the American Statistical Association*, 55(290):299–306, 1960. doi: 10.1080/01621459.1960.10482064.

[13] Christopher D Carroll. Solving microeconomic dynamic stochastic optimization problems. techreport, Johns Hopkins University, 2020. URL https://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs.pdf.

[14] S. Rao Aiyagari. Uninsured Idiosyncratic Risk and Aggregate Saving. *Quarterly Journal of Economics*, 109(3):659–684, 1994. doi: 10.2307/2118417.

[15] Mark Huggett. The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969, 1993. doi: 10.1016/0165-1889(93)90024-m.

[8] Here we can interpret π as the risk premium, that is, the additional average return from holding a risky asset compared to the risk-free rate r. Adjusting the average log return by the asset volatility ensures that increasing $\sigma_{\mathbf{r}}^2$ constitutes a mean-preserving spread of the level of return.

[16] Stephen P. Zeldes. Consumption and Liquidity Constraints: An Empirical Investigation. *Journal of Political Economy*, 97(2):305–346, 1989. doi: 10.1086/261605.

[17] Angus Deaton. Saving and Liquidity Constraints. *Econometrica*, 59(5):1221–1248, 1991. doi: 10.2307/2938366.

[18] Christopher D. Carroll and Miles S. Kimball. On the Concavity of the Consumption Function. *Econometrica*, 64 (4):981–992, 1996. doi: 10.2307/2171853.

[19] Christopher D. Carroll and Patrick Toche. A Tractable Model of Buffer Stock Saving. Working Paper 15265, NBER, 2009.

[20] F. N. Fritsch and R. E. Carlson. Monotone Piecewise Cubic Interpolation. *SIAM Journal on Numerical Analysis*, 17(2):238–246, 1980. doi: 10.1137/0717021.

[21] Karsten O. Chipeniuk. Optimal Grid Selection for the Numerical Solution of Dynamic Stochastic Optimization Problems. *Computational Economics*, 56(4):883–928, 2020. doi: 10.1007/s10614-019-09953-4.

[22] Manuel S. Santos. Accuracy of Numerical Solutions Using the Euler Equation Residuals. *Econometrica*, 68(6): 1377–1402, 2000. doi: 10.1111/1468-0262.00165.

[23] Kenneth L. Judd, Lilia Maliar, and Serguei Maliar. Lower Bounds on Approximation Errors to Numerical Solutions of Dynamic Economic Models. *Econometrica*, 85(3): 991–1020, 2017. doi: 10.3982/ecta12791.

[24] L. M. Benveniste and J. A. Scheinkman. On the Differentiability of the Value Function in Dynamic Models of Economics. *Econometrica*, 47(3):727–732, 1979. doi: 10.2307/1910417.

[25] Paul Milgrom and Ilya Segal. Envelope Theorems for Arbitrary Choice Sets. *Econometrica*, 70(2):583–601, 2002. doi: 10.1111/1468-0262.00296.

[26] Fred N. Fritsch and J. Butland. A Method for Constructing Local Monotone Piecewise Cubic Interpolants. *SIAM Journal on Scientific and Statistical Computing*, 5 (2):300–304, 1984. doi: 10.1137/0905021.

[27] Carl de Boor. *A Practical Guide to Splines*. Springer, revised edition, 2001. doi: 10.1007/978-1-4612-6333-3.

[28] James M. Hyman. Accurate Monotonicity-Preserving Cubic Interpolation. *SIAM Journal on Scientific and Statistical Computing*, 4(4):645–654, 1983. doi: 10.1137/0904045.

[29] Paul A. Samuelson. Lifetime Portfolio Selection by Dynamic Stochastic Programming. *Review of Economics and Statistics*, 51(3):239–246, 1969. doi: 10.2307/1926559.

[30] Robert C. Merton. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics*, 51(3):247–257, 1969. doi: 10.2307/1926560.

[31] Robert C. Merton. Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, 3(4):373–413, 1971. doi: 10.1016/0022-0531(71)90038-X.

[32] Christopher D Carroll. Crra utility with rate risk. techreport, Johns Hopkins University, 2020. Lecture notes.

[33] Jess Benhabib and Alberto Bisin. Skewed Wealth Distributions: Theory and Empirics. *Journal of Economic Literature*, 56(4):1261–1291, 2018. doi: 10.1257/jel.20161390.

[34] Karsten O. Chipeniuk, Nets Hawk Katz, and Todd B. Walker. Households, Auctioneers, and Aggregation. *European Economic Review*, 141:103997, 2021. doi: 10.1016/j.euroecorev.2021.103997.